THEORY OF THE MOTION OF A VORTEX RING UNDER GRAVITY. RISE OF THE CLOUD FROM A NUCLEAR EXPLOSION

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Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 8, No. 2, pp. 3–15, 1967

The motion of a vortex ring under gravity is considered. A system of equations defining the motion is derived, with allowance for turbulent mixing and adiabatic expansion. The case of a small density difference between the ring and the surrounding medium is considered. A numerical solution is presented for an explosion ring in an inhomogeneous atmosphere. Results are compared with experiment and with published theories.

Various natural and artificial processes give rise to vortex rings in which the core differs in density from the surrounding medium. Cloud studies and special experiments have shown that convection resulting from upthrust consists to a large extent of more or less isolated masses of rising air mixing with the surrounding medium. The air within the rising volume circulates much as in a vortex ring [1].

Explosives (including nuclear weapons) produce a fireball that moves upward in the form of a cloud and almost at once becomes a vortex ring. This is clear from photographs [2, 3] and films of nuclear explosions. It is very important to know the motion of the cloud in a nuclear explosion, since the radioactive products represent a hazard to man. A similar effect may occur in a reactor accident.

Here I consider theoretically the motion of a vortex ring under gravity for cases simulating meteorological phenomena (when the density difference between outside and inside of the cloud is small) and explosions (density difference large). The scheme is one proposed by Khristianovich in 1954.

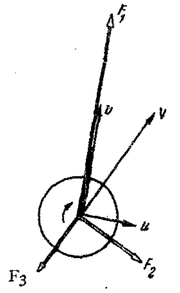

Fig. 1

In an explosion, after the shock wave has escaped, there remains a fireball, which starts to rise when it has ceased to expand. The rising gas is very rapidly transformed to a vortex ring. The maximum upward speed is reached within a few seconds; then the speed falls, and ultimately the gas ceases to rise. The cloud is acted on by wind and disperses by diffusion. The horizontal dimensions of the ring increase as it rises, the rotation ceases, and the cloud acquires a flattened form. If the explosion occurs at the surface, the rise is accompanied by the formation and rise of a dust column, the whole taking on a mushroom-like appearance [4].

The main upward force F_1 is explained by Archimedes' principle (density difference). The friction produced by the motion gives rise to turbulent mixing and vortex circulation. The sphere flattens as it rises, since the pressure at the front and rear critical points is increased relative to the atmospheric pressure, whereas that on the equator is reduced. The rising gas eventually becomes a toroid, within which the air rotates around the horizontal axial line, while outside it forms a circulation flow.

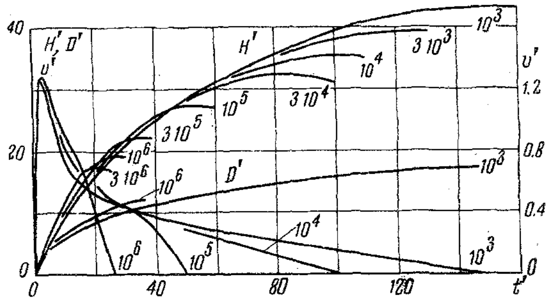

Fig. 2

The circulation gives rise to a force F_2 perpendicular to the direction of motion of a ring element. The horizontal component of this force extends the ring laterally, while the vertical component opposes the rise of the ring.

Because of flow breakaway on the rear side of a ring element, the latter is acted on by a resistance force F_3 directed against the velocity vector (Fig. 1).

The air in the ring is very hot at the start of the rise (it has been taken as 3000° in the calculations), but this is reduced by adiabatic expansion since atmospheric pressure decreases with altitude, and due to turbulence the hot gas mixes with the surrounding cold air. Cooling by thermal emission can be neglected at such temperatures, since the radiation mean free path is much greater than the size of the cloud.

The mixing of the core with the surrounding air reduces the angular velocity of the air in the core, and this involves a reduction in the circulation of the external flow, which reduces F_2. The horizontal speed of a ring element is reduced by the friction produced by the outside air.

Table 1

x	$v'/\sqrt{\Delta_0}$	$u'/\sqrt{\Delta_0}$	R'/R_0'	r'/r_0'	H'	$t'\sqrt{\Delta_0}$	n	k_1
1.0	0.19	0	1.0	1.0	0	0		
0.98	0.362	0.014	1.002		0.111	0.41		
0.90	0.708	0.068	1.04		0.597	1.31		
0.80	0.847	0.121	1.16	1.04	1.26	2.16		
0.70	0.900	0.165	1.32	1.04	2.02	3.03		
0.50	0.809	0.216	1.90	1.03	3.93	5.27		
0.30	0.625	0.219	3.14	1.03	6.81	9.35	2.95	
0.10	0.382	0.168	6.95	1.20	14.5	23.4	3.10	
0.06	0.311	0.139	9.30	1.34	18.2	33.7	2.96	
0.04	0.264	0.118	11.3	1.50	21.3	44.5	2.89	0.092
0.02	0.210	0.087	14.8	1.84	27.8	72.4	2.90	
0.01	0.166	0.061	18.8	2.31	36.0	117	2.96	0.091
0.006	0.140	0.046	22.1	2.74	43.1	165	3.01	0.082

The following assumptions are made in discussing the rise of a vortex ring:

1. The pressure within the ring equals the outside air pressure, and adiabatic expansion occurs during ascent.

2. All quantities within the ring (density, temperature, speed, etc.) are identical throughout the cross section, and any changes with height occur instantaneously throughout the ring.

Table 2

q TH	$\max H'^{+}$	$D'/\max H'^{+}$	
		$t' = t'^{+}$	$t' = 200$
10^3	46	0.39	0.41
10^4	38	0.42	0.49
10^5	29.4	0.45	0.62
10^6	21.7	0.51	0.87

3. Turbulent mixing of the hot air with the cold outer air occurs at the surface of the ring, as in a turbulent jet.

4. The motion of an element in a vortex ring is considered as the motion of a circular cylinder.

5. The pressure and temperature in the atmosphere are distributed in accordance with the International Standard Atmosphere and with the results of measurements for great heights [5].

The results can be used in discussing the motion of the cloud formed in an explosion.

The effects are qualitatively the same as in the fall of a small volume of a liquid of slightly different density [1]. The experimental results will be used to determine the values of the parameters governing the equations describing the motion of a vortex ring.

Table 3

q TH	$\max H'$	H_i'	$\max H'/H_i'$	ΔH km
10^3	44	38.3	1.15	0.57
10^4	36	31.1	1.16	1.05
10^5	27.4	24.2	1.13	1.48
$3 \cdot 10^5$	22.8	19.4	1.17	2.38
10^6	19.7	15.7	1.25	4

NOTATION

ρ, p, T—density, pressure, and temperature, respectively, of the air in the ring;

ρ_1, p_1, T_1—same for the atmosphere;

ζ—ratio of air densities inside and outside the ring;

θ—potential temperature;

R—radius of the axial line of the toroid;

r—radius of cross-section of toroid;

W—volume of toroid;

S—area of the side surface of a ring element;

S_1—area of the middle section of a ring element;

M—mass of air in the vortex ring;

M_1—additional mass of the toroid in transient motion;

v—rate of ascent of the toroid;

u—speed of horizontal motion of a ring element;

V—speed of ring element;

C_x—resistance coefficient;

α—entertainment factor in turbulent mixing;

Γ—circulation of external flow around the ring;

v_i—inductive velocity of ring (steady-state velocity of a ring unperturbed by external forces);

g—acceleration due to gravity;

q—TNT equivalent (tons) of explosion;

a_0—radius of the fireball at start of rise of cloud;

H—height of rise of the axial line of the ring;

H^{+}—height of rise of the upper edge of the ring;

d—vertical dimension of the ring;

D—horizontal dimension of the ring;

t^{+}—time to attain maximum height;

$\varkappa$—ratio of specific heats;

$$\zeta = \frac{\rho}{\rho_1} = \frac{T_1}{T}, \quad \theta = T_1\left[\frac{p_{01}}{p_1}\right]^{\frac{\varkappa-1}{\varkappa}}, \quad H^{+} = H + r,$$

$$D = 2(R + r), \quad \varkappa = \frac{c_p}{c_v}, \quad M_1 \approx \rho_1 W, \quad \mathrm{Re} = \frac{2rV}{\nu}.$$

§ 1. DERIVATION OF EQUATIONS

Equation of motion. The rate of change of momentum for the ring and surrounding air (the latter incorporated via M_1) is equal to the sum of F_1, F_2, and F_3;

$$F_2 = \rho_1 V\Gamma R d\varphi, \quad F_3 = {}^1/_2 C_x \rho_1 V^2 S_1.$$

Projection of the vector equation on the horizontal and vertical directions gives us the equations of mo-

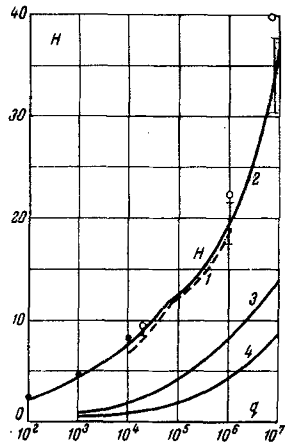

Fig. 3

tion of the ring:

a) vertical projection

$$\frac{d}{dt}[v(M + M_1)] =$$

$$= gW\rho_1(1-\zeta) - 2\pi\rho_1\Gamma R u - \frac{C_x S_1}{2}\rho_1 v V; \qquad (1.1)$$

b) horizontal projection

$$\frac{d}{dt}[u(M + M_1)] = 2\pi\rho_1\Gamma v R - \frac{C_x S_1}{2}\rho_1 u V. \qquad (1.2)$$

Heat-balance equation. Consider the heat balance while the height of the cloud changes by dH [8]. The air temperature is $T(p)$ at a height where the pressure is p and the cloud has mass M. The pressure is $p + dp$ at height $H + dH$. If there were no mixing with the surrounding air, the temperature in the cloud would become

$$T(p)\left[1 + \frac{\varkappa - 1}{\varkappa}\frac{dp}{p}\right]$$

owing to adiabatic expansion; but the cloud entrains a mass dM during this time. We assign all the entrained mass to the level $p + dp$. The total heat content of the mixed mass should be equal to the sum of the heat contents of the initial and entrained masses. We neglect quantities of the second order of smallness and also the difference in the values of the specific heats to get

$$\frac{dT}{T}=\frac{dT_1}{T_1}-\frac{d\zeta}{\zeta}=\frac{\varkappa-1}{\varkappa}\frac{dp}{p}-(1-\zeta)\frac{dM}{M}. \qquad (1.3)$$

Mixing with cold air. Turbulent mixing occurs at the surface, which is considered via a relation between the entrainment rate, the ring speed, and the air densities. This relation takes a form similar to that for mixing of a gas jet with a medium of a different density [9,10]. If V is the speed at the axis of the jet, where the air density is ρ, and if the density of the surrounding air is ρ_1, the entrainment rate is $v^\circ = \alpha V(\rho/\rho_1)^{1/2}$. This gives us the rate of mass entrainment in the area S as

$$dM/dt = \alpha VS(\rho\rho_1)^{1/2}.$$

In our problem we assume that

$$\frac{1}{M}\frac{dM}{dt}=\frac{2\alpha V}{r\sqrt{\zeta}}. \qquad (1.4)$$

Circulation equation. Γ is determined by the angular velocity within the toroid, which decreases due to entrainment; discrepancy between Γ and the angular velocity causes eddy breakaway, which alters Γ.

The angular momentum of an element of length $Rd\varphi$ is $((M\Gamma/\pi)/4)(\pi^1 d\varphi)/2)$, on the assumption of constant angular velocity in the toroid. This is altered by the frictional torque, which depends on the momentum transfer by mixing. We assume a linear relation between Γ and the angular velocity, so that torque is dependent on the entrainment rate, on V, and on Γ. Dimensional arguments give

$$\text{const}\frac{dM}{dt}\Gamma = \text{const}\, 2\pi\alpha V\sqrt{\rho\rho_1}Rr\Gamma d\varphi,$$

which resembles the frictional torque for a rotating solid cylinder in a flow of liquid [11], so the equation for Γ is

$$d\Gamma/dt = -\beta 2\alpha\Gamma V/r\sqrt{\zeta}. \qquad (1.5)$$

The constant is to be determined by comparison with experiment. To (1.1)–(1.5) we add

$$u = dR/dt, \quad v + v_i = dH/dt, \qquad (1.6)$$

$$M = \rho_1\zeta 2\pi^2 r^2 R,$$

$$v_i \approx \frac{\Gamma}{4\pi R}\left\{\ln\frac{8R}{r}-\frac{1}{4}-\frac{12\ln(8R/r)-15}{32}\left(\frac{r}{R}\right)^2\right\}. \qquad (1.7)$$

This gives us a system of equations for v, u, R, r, H, Γ, and ζ, which characterize the motion of a vortex ring.

Initial conditions. The equations are to be integrated from t_0, the time when the vortex ring forms, when $v = v_0$ and $\Gamma = \Gamma_0$ can be related by equating $v_0 + v_i$ to the velocity induced by a vortex filament of radius R_0 on the plane of the ring:

$$v_0 = \Gamma_0/2R_0 - v_i \approx 0.64\ \Gamma_0/a_0.$$

At t_0 we also have $\zeta_0 = 0.1$, $u_0 = 0$, and equal volumes for sphere and toroid, so $R_0 = r_0 = 0.596a_0$. We assume that up to time t_0 the cloud rises only in response to gravity and that ζ remains constant during this time, while M_1 equals the value for the toroid, and so

$$t_0 = 0.78\gamma_0\sqrt{a_0/g}, \quad H_0 = 0.25\gamma_0^2 a_0,$$

$$\gamma_0 = \Gamma_0/a_0\sqrt{a_0 g},$$

in which γ_0 is the dimensionless circulation.

The equations and initial conditions can be put in dimensionless form, with the following characteristic quantities: a_0 for length, $(a_0 g)^{1/2}$ for velocity, and $(a_0/g)^{1/2}$ for time. Then

$$R' = \frac{R}{a_0}, \quad v' = \frac{v}{\sqrt{a_0 g}},$$

$$t' = \frac{t}{\sqrt{a_0/g}}, \quad \gamma = \frac{\Gamma}{a_0\sqrt{a_0 g}} \text{ etc.}$$

This a_0 enters the equations as a coefficient for $d\ln\theta/dH$, whose magnitude determines the stability of the stratification in the atmosphere; if this is zero (homogeneous or adiabatic atmosphere), the phenomena would have scale similarity, but this is not a realistic case, since then max $H \to \infty$. If $d\ln\theta/dH > 0$, max H is finite, but there is no similarity. The larger a_0, the more important the terms containing gradients in p_1, ρ_1, and T_1. These terms are small at the start of the motion, so approximate similarity with respect to a_0 should exist at that stage. These terms are most important at the end of the ascent and define max H.

The solution is dependent on α, β, and γ_0. Experiments on turbulent mixing of jets [12,13] show that α lies in the range 0.03 to 0.08; the value $\alpha = 0.055$ was used here. The order of γ_0 can be determined by considering the initial motion of a hot sphere.

Consider an elementary sector. The momentum relative to the axis of the vortex will be affected by the surface friction. We take the shape of the volume and the speed of the external flow as corresponding to t_0, since the speed is greatest at this instant, and so the friction makes the greatest contribution to producing the circulation. We use (1.5) and the relation between v_0 and γ_0 to get

$$\gamma_0 = \left[\frac{3}{8\pi 4.4(0.64)^2\alpha}\right]^{1/2}\left(\frac{(1-\zeta)\sqrt{\zeta}}{1+\zeta}\right)^{1/2}.$$

Now $\gamma_0 = 0.55$ for $\zeta = 0.1$, which corresponds to a rotational speed of several m/sec for the air in the core, which is quite reasonable. We adopt the above form for the relation between γ_0 and ζ, with the constant to be determined by experiment:

$$\gamma_0 = \gamma^* \left(\frac{1-\zeta}{1+\zeta}\right)^{1/2} \zeta^{1/4}. \tag{1.8}$$

If the density difference is small, the result agrees with that of Scorer [1]. The value for β is also to be determined from experiment; $\beta = +1$ corresponds to constant momentum, while $\beta = 0$ corresponds to constant circulation.

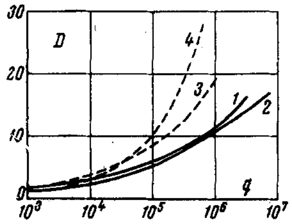

Fig. 4

We therefore have the following system of equations:

$$\frac{dv'}{dt'} = \frac{1-\zeta}{1+\zeta} - \frac{\gamma u'}{\pi(1+\zeta)(r')^2} - \frac{v'V'}{r'}\left[\frac{C_x}{\pi(1+\zeta)} + \frac{2\alpha}{\sqrt{\zeta}}\right] + $$

$$+ \frac{v'}{(1+\zeta)\zeta}\frac{d\zeta}{dt'}, \qquad \frac{du'}{dt'} = \frac{\gamma v'}{\pi(1+\zeta)(r')^2} -$$

$$- \frac{u'V'}{r'}\left[\frac{C_x}{\pi(1+\zeta)} + \frac{2\alpha}{\sqrt{\zeta}}\right] + \frac{u'}{(1+\zeta)\zeta}\frac{d\zeta}{dt'},$$

$$\frac{d\gamma}{dt'} = -\frac{2\alpha\beta\gamma V'}{r'\sqrt{\zeta}},$$

$$\frac{dr'}{dt'} = \frac{r'}{2}\left[\frac{2\alpha V'}{r'\sqrt{\zeta}} - \frac{1}{\zeta}\frac{d\zeta}{dt'} - \frac{u'}{R'} - v'\frac{d\ln\rho_1}{dH}a_0\right],$$

$$\frac{d\zeta}{dt'} = \zeta\left[\frac{2\alpha(1-\zeta)V'}{r'\sqrt{\zeta}} + v'a_0\frac{d\ln\theta}{dH}\right],$$

$$\frac{dH'}{dt'} = v' + v_i', \quad \frac{dR'}{dt'} = u', \quad V' = \sqrt{(u')^2 + (v')^2},$$

$$v_i' = \frac{\gamma}{4\pi R'}\left[\ln\frac{8R'}{r'} - \frac{1}{4} - \frac{12\ln(8R'/r') - 15}{32}\left(\frac{r'}{R'}\right)^2\right] \tag{1.9}$$

with the initial conditions

$$t_0' = 0.78\gamma_0, \quad H_0' = 0.25\gamma_0^2 + h_0', \quad v_0' = 0.64\gamma_0,$$
$$u_0' = 0, \quad R_0' = r_0' = 0.596, \quad \zeta_0 = 0.1.$$

§ 2. SMALL DENSITY DIFFERENCE. RESULTS OF [1]

A solution of greater density is dropped from a cup into a vessel containing water; this solution is rapidly transformed to a vortex ring. After the ring had traveled a distance roughly equal to half the diameter of the cup (diameter 7.62 cm), the motion became slower and measurements were made on this part of the motion. During this time, the density difference in most cases was less than 5%, and by the end of the motion had fallen to 0.1%. A distance of 107 cm was covered in 5–30 sec, in accordance with the density difference. The motion was entirely turbulent and was independent of the viscosity. The following relationships were derived and confirmed by experiment:

$$k_1 (H')^2 = t'(1-\zeta_0)^{1/2},$$

$$H'^+ = {}^1/_2 D'n, \quad v = C\,[{}^1/_2 gD(1-\zeta)]^{1/2}. \tag{2.1}$$

Here the constant $k_1 \approx 0.09$, while n remains constant during the motion (it was found that n increased systematically with $1 - \zeta_0$), and $C = 1.2$.

Solution of the equations. The general equations give the following for a homogeneous liquid in a surrounding medium:

$$W(1-\zeta) = \text{const}, \qquad \gamma = \gamma_0\left[\frac{(1-\zeta)\zeta_0}{(1-\zeta_0)\zeta}\right]^{\beta}. \tag{2.2}$$

These are used with (1.8) and the symbols $1 - \zeta = \Delta$ and $\Delta/\Delta_0 = x$ to write the equations in the form

$$\frac{d}{dx}\frac{V'^2}{\Delta_0} = \frac{1}{(2-\Delta_0 x)x\sqrt{1-\Delta_0 x}} \times$$

$$\times \left\{\frac{V'^2}{\Delta_0}\left[\frac{C_x}{\pi\alpha} + 4\sqrt{1-\Delta_0 x}\right] - \frac{v'}{V'}\frac{r'}{r_0'}x\frac{r_0'}{\alpha}\right\}, \tag{2.3}$$

$$\frac{d}{dx}\frac{u'}{\sqrt{\Delta_0}} = \frac{1}{2(2-\Delta_0 x)x\sqrt{1-\Delta_0 x}} \times$$

$$\times \left\{\frac{u'}{\sqrt{\Delta_0}}\left[\frac{C_x}{\pi\alpha} + 4\sqrt{1-\Delta_0 x}\right] - \right.$$

$$\left. - \frac{\gamma^*(1-\Delta_0)^{\beta+1/4}}{\pi\alpha r_0'(2-\Delta_0)^{1/2}}\frac{r_0' v'}{r'V'}\frac{x^\beta}{(1-\Delta_0 x)^\beta}\right\}, \tag{2.4}$$

$$\frac{2}{3}\frac{d}{dx}\left(\frac{r_0'}{r'}\right)^3 = \frac{1}{x}\left(\frac{r_0'}{r'}\right)^3 - \frac{1}{2\alpha\sqrt{1-\Delta_0 x}}\frac{u'}{V'}, \tag{2.5}$$

$$\frac{dH'}{dx} = -\frac{r_0'}{2\alpha x\sqrt{1-\Delta_0 x}}\frac{r'}{r_0'}\frac{v'}{V'},$$

$$\frac{d}{dx}[t'\sqrt{\Delta_0}] = -\frac{r_0'}{2\alpha x\sqrt{1-\Delta_0 x}}\frac{\sqrt{\Delta_0}}{V'}\frac{r'}{r_0'},$$

$$\frac{R'}{R_0'} = \left(\frac{r_0'}{r'}\right)^2\frac{1}{x} \tag{2.6}$$

and the initial conditions as

$$x = 1, \quad \frac{R'}{R_0'} = \frac{r'}{r_0'} = 1,$$

$$H' = t' = u' = 0, \quad \frac{v'}{\Delta_0^{1/2}} = 0.64\frac{(1-\Delta_0)^{1/4}}{(2-\Delta_0)^{1/2}}.$$

The approximate behavior for $x \to 0$ is considered in order to determine the dependence on the parameters. The left parts in (2.3)–(2.5) are neglected (numerical estimates show that this is justified), and it is also assumed that $v' = V'$. Then

$$\frac{r'}{r_0'} \to \left(\frac{B}{x^{1+2\beta}}\right)^{1/3}, \qquad \frac{R'}{R_0'} \to \left(\frac{1}{B^2 x^{1-4\beta}}\right)^{1/3},$$

$$\frac{v'}{\Delta_0^{1/2}} \to \left[\frac{\pi r_0'}{C_x + 4\pi\alpha}\right]^{1/2}\left(\frac{\sqrt{B}}{x^{1-\beta}}\right)^{1/3};$$

$$B = \frac{4\pi\alpha^2(r_0')^3(2-\Delta_0)(C_x + 4\pi\alpha)}{(\gamma^*)^2(1-\Delta_0)^{1/2(1+4\beta)}}.$$

We use these relations to integrate the following two equations:

$$H' \to \frac{3r_0'}{2\alpha(1+2\beta)}\left(\frac{B}{x^{1+2\beta}}\right)^{1/3},$$

$$t'\Delta_0^{1/2} \to \frac{3r_0'}{2\alpha(2+\beta)}\left(\frac{\sqrt{B}}{x^{2+\beta}}\right)^{1/3}\left(\frac{C_x+4\pi\alpha}{\pi r_0'}\right)^{1/2}.$$

Here H' is only slightly dependent on α. For n we get

$$n = \frac{H'+r'}{R'+r'} = \frac{6\pi\alpha(r_0')^3(C_x+4\pi\alpha)(2-\Delta_0)}{(1+2\beta)\gamma^*(1-\Delta_0)^{1/2(1+4\beta)}x^{2\beta}}.$$

Thus n increases as x decreases, but the main motion of the ring involves only a fairly small change in x near 0.01 (as found by numerical calculation), so n is almost constant over much of the path. The dependence of n on Δ_0 is important. If n_0 is the value corresponding to $\Delta_0 = 0$, we have

$$\frac{n}{n_0} = \frac{2-\Delta_0}{2(1-\Delta_0)^{1/2(1+4\beta)}},$$

i.e., n/n_0 increases monotonically with Δ_0. Also, $t'(\Delta_0)^{1/2}(H')^{-2}$ is not constant for $\beta \neq 0$, but it varies little over much of the path.

The case $\beta = 0$ corresponds to γ = constant, when (2.1) is strictly obtained, but n does not depend on Δ_0. There is little change in Γ over much of the path over which measurements are made.

The case $\beta = 1$ corresponds to conservation of momentum, when n varies too greatly with Δ_0. Calculations for $\Delta_0 \to 0$ have been compared with experiment to give $\beta = +0.2$ and $\gamma^* = 0.4$. Table 1 gives the numerical solution for these values of the parameters. More detailed data are needed in order to obtain more accurate values of the parameters, but only $\Delta_0 W_0$ is given by Scorer [1]. If it is assumed that the volume of the liquid equals that of the cup, then $a_0 = 3$ cm. Calculation shows that the motion of the ring becomes slower after a distance of 9 cm (1.2 times the cup diameter), while experiment gave this distance as 1.5 times the diameter. The solution to the equations gives $v = 1.3\,(gr\,\Delta)^{1/2}$, which agrees well with (2.3). The time taken to move a distance of 107 cm was

$$\frac{6}{\sqrt{\Delta_0}}\left[\frac{1-\Delta_0/2}{(1-\Delta_0)^{0.9}}\right]^{1/6},$$

which gives 10–60 sec for Δ_0 between 0.4 and 0.01.

These results show that this scheme gives a good description of the observations. Values of β and γ^* were derived from those experiments that can be applied to large rings with substantial density differences. Here allowance must be made for the dependence of the resistance coefficient on the Reynolds number. The model experiments employed $C_x = 1.2$, since the number R was 10^3–10^5, whereas $C_x = 0.4$ for the cloud from a nuclear explosion, since $R > 10^6$ [12].

§3. SPHERE RADIUS AS A FUNCTION OF EXPLOSION ENERGY

The heat energy received by the gas in the sphere can be calculated from results for a point explosion with allowance for the counterpressure [14]. Approximately one-third of the energy liberated by the explosion is emitted in the form of radiation [15]. The energy in the sphere at the start of the ascent is about one-fourth of the total explosion energy. The volume is

$$\frac{4\pi a_0^3 p_0}{3(\varkappa-1)} = 0.25E_0\,.$$

The radius a_0 [m] is given [16] as a function of the TNT equivalent q by

$$a_0 = \left(\frac{p_{00}}{p_0}\right)^{1/3} q^{1/3}\cdot 10\,, \tag{3.1}$$

in which p_{00} is the atmospheric pressure at sea level (760 mm Hg) and p_0 is the pressure at the detonation height h_0.

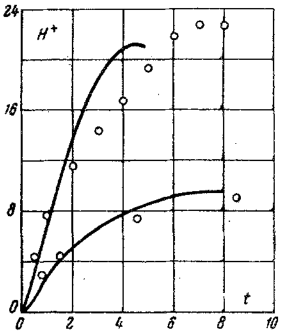

Fig. 5

§4. RESULTS FOR THE STANDARD ATMOSPHERE

Figure 2 gives the t' dependence of $H' - h_0'$, D', and v' for various q at $h_0' = 0$. Conversion to dimensional terms should be made via the above expressions and the dependence of a_0 on q. The solutions for various q are not similar because the atmosphere is not vertically homogeneous. The dimensionless height of ascent decreases as a_0 increases because a given H' corresponds to greater heights, and hence to lower pressures and greater adiabatic expansion, so the upward force for large a_0 decreases more rapidly as H' increases, and max H' is reduced. The height of ascent in km increases with q; Fig. 3 shows max H as a function of q. The curves of Fig. 2 allow one to judge the similarity in the behavior of H', D', and v' as functions of t' in the initial part of the motion. The similarity tends to be lost as t' increases, and it applies to a narrower range of q.

A notable feature is the increase in lateral spread with q after attainment of max H. An element of the ring loses horizontal momentum less rapidly as it enters layers of lower density, and the lateral spread becomes more pronounced. Table 2 gives results on this spread.

The max H for the center of the cloud exceeds the level at which the densities inside and outside the cloud are equal (equilibrium level H_1), and this excess height $\Delta H = H - H_1$ increases with q (Table 3).

The oscillations of these large masses of air set up internal waves in the atmosphere [17,18]. The inertia carries the cloud beyond H_1, which leads to damped oscillations about H_1 with a period

$$\tau = \frac{2\pi}{[{}^1\!/_2 gd\ln\theta/dH]^{1/2}}\ [\mathrm{sec}]\,.$$

This formula is derived from the equations for the motion near H_1, in which we can take v' and $1 - \zeta$ as small and neglect the lateral expansion. Then τ is dependent on H_1, and the dependence for the atmosphere used in the calculations is as follows:

$H_1 =$ 0–11	11–25	30	40	45	km
$\tau =$ 780–750	417	380	405	415	sec.

The time t_+ taken to reach max H varies as follows with q:

$q =$ 10^3	10^4	10^5	10^6	tons
$t_+ =$ 480	465	340	260	sec.

Approximate relation for max H. We assume that $v' = V$ and $2\alpha/r' = \text{const}$ on the basis of numerical calculations, which show that the first assumption is obeyed closely. Then the equations for H' and ζ give

$$\frac{dH'}{d\zeta} = \left[(1-\zeta)\,\zeta^{1/2}\frac{2\alpha}{r'} + \zeta a_0 \frac{d\ln\theta}{dH}\right]^{-1},$$

the solution being

$$H' \approx \frac{r'}{2\alpha}\ln\left[\frac{(1+\sqrt{\zeta})(1-\sqrt{\zeta_0})}{(1-\sqrt{\zeta}-{}^1/_4 a_0 r'\alpha^{-1} d\ln\theta/dH)(1+\sqrt{\zeta_0})}\right],$$

since we can neglect $1/4\, a_0 r'\alpha^{-1} d\ln\theta/dH$ relative to unity. Although the exact solution indicates that the cloud reaches max H for $\zeta > 1$ and oscillates about H_1, we assume that $\zeta_{max} = 1$. Then the approximate expression for the maximum height reached by the center of the cloud as a function of the size of the fireball is as follows:

$$H'_{max} \approx \frac{r'}{2\alpha}\ln\left[\frac{(1-\sqrt{\zeta_0})\,8\alpha}{(1+\sqrt{\zeta_0})\,r' a_0 d\ln\theta/dH}\right]. \qquad (4.1)$$

An atmosphere with more stable stratification (larger $d\ln\theta/dH$) gives a lower height. The dependence on ζ_0 is not very pronounced.

However, ζ_{max} does differ from unity (and the more so, the greater q), so we should use a relation in which the coefficients are determined by numerical calculation:

$$H_{max} \approx 11 a_0 \ln\frac{5750}{a_0}. \qquad (4.2)$$

§ 5. DUST COLUMN

A nuclear explosion produces a layer of dust near the ground and a rising dust column at the epicenter [4]. The size and type of motion of the column are dependent on q and h_0. Here the rise of the top of the dust column is discussed qualitatively.

The upper layer of the ground is vaporized by the thermal radiation, and the resulting layer of dust-laden air is further heated by the radiation. The temperature distribution in this layer peaks at the epicenter. The temperature at the epicenter and the radial rate of decrease are dependent on q and h_0. This gives a rising current at the epicenter, and hence, a dust column. The air gradually cools by expansion and turbulent mixing, and the rate of rise decreases. An explosion at ground level means that the rise of the fireball affects that of the dust column, but this effect is small if $h_0 > a_0$.

The motion of the head of the column may be considered to a first approximation in isolation from that of the rest of the column as the rise of a sphere of hot air of a certain radius (this to be found by experiment or calculation), since the air at the top of the column is hottest and runs ahead of the rest of the column. The heating is inversely dependent on h_0 for q = const. Temperatures measured as a function of h_0 may be used as initial data in integrating a system of equations similar to that describing the rise of the fireball. This shows that the column will catch up with the cloud for $h_0 > h_0^*$. Photographs show [2,3] that the dust column does not link up with the fireball for a long time.

§ 6. COMPARISON WITH PUBLISHED DATA

The literature [19–23] gives two methods for determining H_{max}. Sutton and Machta assume that the rise of the fireball is produced by F_1, while cooling occurs by adiabatic expansion and mixing with the surrounding air, inertia and circulation in the cloud being neglected. These studies deal only with the maximum height reached, the equilibrium level being taken as that height. Morton et al. [24] also consider the cloud, with allowance for oscillations as well as cooling, but without incorporation of the circulation. Sutton gives

$$\max H = 0.665\ q^{0.276}\ \text{km}.$$

This formula can be used up to 11 km, since it assumes a linear dependence of the potential temperature on the height, which is so for this range of heights. The results agree well with Sutton's formula for q from 0.1 to 10^3 tons. Machta gives

$$\max H =$$

$$= \left(\frac{1}{M}\frac{dM}{dH}\right)^{-1}\ln\left\{\frac{1}{d\theta/dH}\frac{1}{M}\frac{dM}{dH}\left[(\Delta\theta)_0 + \left(\frac{1}{M}\frac{dM}{dH}\right)^{-1}\frac{d\theta}{dH}\right]\right\}.$$

This formula has the structure of the approximate formula (4.1), but Machta does not give an explicit dependence on q, and so he erroneously states that the application of the formula is restricted. This error has been corrected [25], and the q-dependence has been given for $5 \cdot 10^3$ to 10^6 tons for various distributions of the potential temperature (tropics, middle latitudes), as well as experimental evidence on the diameter and vertical size as functions of q. It is stated that the cloud spreads sideways after reaching max H to an extent that increases with q, and also that the cloud ceases to rise after 4–6 min.

Figure 3 compares the results for max H derived from Sutton's formula (filled circles) with Machta's curve 1 and with curve 2 derived here; the open circles are experimental values [15, 26]. The present results agree well with those of [25] for q of 10^3 to $3 \cdot 10^6$ tons. The vertical lines at q of $2 \cdot 10^4$, 10^6, and $8 \cdot 10^6$ tons indicate the vertical dimensions of the cloud. Observations qualitatively confirm the predicted change in vertical size with q (curve 4), but the values are only half the observed ones [25] (curves 3). The discrepancy may arise because the calculation scheme is oversimplified, in particular by assuming uniformity of all quantities throughout the cloud.

Figure 4 shows the horizontal size D (km) [25] (curve 2) at the instant when max H is reached, whereas calculation for lateral expansion (curve 3) gives a different result (curve 4). The following results have been published [27]: a thermonuclear device of yield $8 \cdot 10^6$ tons (on 1 November 1952) gave a cloud that rose to a height of 25 miles (40 km). The calculations indicate that $q = 10^7$ tons would cause the center of the cloud to rise to 37 km.

Figure 5 shows $H^+(t)$ (height in km, t in min) for q of $2 \cdot 10^4$ and 10^6 tons from calculation (solid lines) and experiment [15, 26], i.e., the results for max H for q from 0.1 to 10^7 tons agree satisfactorily with published values.

I am indebted to S. A. Khristianovich for direction in this work and to V. I. Kozhevnikov for performing the computations.

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